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The nonlinear Schrödinger equation on the interval

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Abstract

Let $q(x, t)$ satisfy the Dirichlet initial-boundary value problem for the nonlinear Schrödinger equation on the finite interval, $0 < x < L$, with $q_0(x) = q(x, 0)$, $g_0(t) = q(0, t)$, $f_0(t) = q(L, t)$. Let $g_1(t)$ and $f_1(t)$ denote the *unknown* boundary values $q_x(0, t)$ and $q_x(L, t)$, respectively. We first show that these unknown functions can be expressed in terms of the given initial and boundary conditions through the solution of a system of nonlinear ODEs. For the focusing case it can be shown that this system has a global solution. It appears that this is the first time in the literature that such a characterization is explicitly described for a nonlinear evolution PDE defined on the interval; this result is the extension of the analogous result of [4] and [20] from the half-line to the interval. We then show that $q(x, t)$ can be expressed in terms of the solution of a 2×2 matrix Riemann–Hilbert problem formulated in the complex k-plane. This problem has explicit (x, t) dependence in the form $\exp[2ikx + 4ik^2t]$, and it has jumps across the real and imaginary axes. The relevant jump matrices are explicitly given in terms of the spectral data $\{a(k), b(k)\}$, $\{A(k), B(k)\}$, and $\{\mathcal{A}(k), \mathcal{B}(k)\}$, which in turn are defined in terms of $q_0(x)$, $\{g_0(t), g_1(t)\}$, and $\{f_0(t), f_1(t)\}$, respectively.

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1. Introduction

We analyse the Dirichlet initial-boundary value problem for the nonlinear Schrödinger (NLS) equation on a finite interval:

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} - 2\lambda|q|^2 q = 0 && \lambda = \pm 1 \quad 0 < x < L \quad 0 < t < T \\ &q(x, 0) = q_0(x) && 0 < x < L \\ &q(0, t) = g_0(t) \qquad q(L, t) = f_0(t) && 0 < t < T \end{aligned} \tag{1.1}$$

where L, T are positive constants and q_0, g_0, f_0 are smooth functions compatible at $x = t = 0$ and at $x = L, t = 0$, i.e. $q_0(0) = g_0(0), q_0(L) = f_0(0)$.

Our analysis is based on the extension of the results of [4], [6] and [20] from the half-line to the interval.

The analysis involves three steps.

Step 1. A RH formulation under the assumption of existence. We *assume* that there exists a smooth solution $q(x, t)$.

We use the *simultaneous* spectral analysis of the associated Lax pair of the NLS to express $q(x, t)$ in terms of the solution of a (2×2)-matrix Riemann–Hilbert (RH) problem defined in the complex k-plane. This problem has *explicit* (x, t) dependence in the form of $\exp\{2\mathrm{i}(kx + 2k^2 t)\}$, and it is uniquely defined in terms of the so-called *spectral functions*,

$$\{a(k), b(k)\} \qquad \{A(k), B(k)\} \qquad \{\mathcal{A}(k), \mathcal{B}(k)\}. \tag{1.2}$$

These functions are defined in terms of

$$q_0(x) \qquad \{g_0(t), g_1(t)\} \qquad \{f_0(t), f_1(t)\} \tag{1.3}$$

respectively, where $g_1(t)$ and $f_1(t)$ denote the *unknown* boundary values $q_x(0, t)$ and $q_x(L, t)$.

We show that the spectral functions (1.2) are *not* independent but they satisfy the *global relation*

$$(a\mathcal{A} + \lambda\bar{b}e^{2\mathrm{i}kL}\mathcal{B})B - (b\mathcal{A} + \bar{a}\,\mathrm{e}^{2\mathrm{i}kL}\mathcal{B})A = \mathrm{e}^{4\mathrm{i}k^2 T}c^+(k) \qquad k \in \mathbb{C} \tag{1.4}$$

where $c^+(k)$ is an entire function which is of $O(1/k)$ as $\to \infty$, $\operatorname{Im} k > 0$; in fact,

$$c^+(k) = O\left(\frac{1 + \mathrm{e}^{2\mathrm{i}kL}}{k}\right) \qquad k \to \infty.$$

Step 2. Existence under the assumption that the spectral functions satisfy the global relation. Motivated from the results of step 1, we *define* the spectral functions (1.2) in terms of the smooth functions (1.3). We also *define* $q(x, t)$ in terms of the solution of the RH problem formulated in step 1. We *assume* that there exist smooth functions $g_1(t)$ and $f_1(t)$ such that the spectral functions (1.2) satisfy the global relation (1.4). We then prove that (i) $q(x, t)$ is defined globally for all $0 < x < L, 0 < t < T$; (ii) $q(x, t)$ solves the NLS equation; (iii) $q(x, t)$ satisfies the given initial and boundary conditions, i.e. $q(x, 0) = q_0(x), q(0, t) = g_0(t)$, $q(L, t) = f_0(t)$. A byproduct of this proof is that $q_x(0, t) = g_1(t)$ and $q_x(L, t) = f_1(t)$.

Step 3. The analysis of the global relation. Given q_0, g_0, f_0, we show that the global relation (1.4) characterizes g_1 and f_1 through the solution of a system of nonlinear Volterra integral equations. For the focusing case using the results of [20], it can be shown that there exists a global solution. For the defocusing case the rigorous investigation of these equations remains open.

We now discuss further the above three steps.

The analysis of step 1 is based on the introduction of appropriate eigenfunctions which satisfy *both* parts of the Lax pair. It was shown in [1] that for linear PDEs defined in a polygonal domain with N corners, there exists a canonical way of choosing such eigenfunctions: there exist N such eigenfunctions each of them normalized with respect to each corner. Motivated by this result we introduce four eigenfunctions, $\{\mu_j(x, t, k)\}_1^4$, see figure 1, such that

$$\mu_1(0, T, k) = I \qquad \mu_2(0, 0, k) = I \qquad \mu_3(L, 0, k) = I \qquad \mu_4(L, T, k) = I \tag{1.5}$$

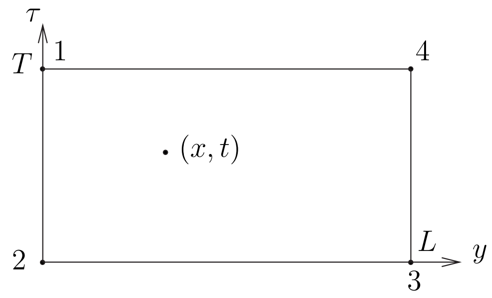

Figure 1. The normalization points of $\{\mu_j(x_1 t_1 k)\}_{j=1}^4$.

where μ_j are 2×2 matrices and $I = \mathrm{diag}(1,1)$. It can be shown that these eigenfunctions are simply related through the three matrices s, S, S_L

$$
\begin{aligned}
&s(k) = \mu_3(0,0,k),\ S(k) = \left(\mathrm{e}^{2\mathrm{i}k^2T\sigma_3}\mu_2(0,T,k)\,\mathrm{e}^{-2\mathrm{i}k^2T\sigma_3}\right)^{-1}\\
&S_L(k) = \left(\mathrm{e}^{2\mathrm{i}k^2T\sigma_3}\mu_3(L,T,k)\,\mathrm{e}^{-2\mathrm{i}k^2T\sigma_3}\right)^{-1}
\end{aligned}
\tag{1.6}
$$

where $\sigma_3 = \mathrm{diag}(1,-1)$. These matrices satisfy certain symmetry properties, thus they can be represented by

$$
s(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \overline{\lambda b(\bar{k})} & a(k) \end{pmatrix} \qquad S(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \overline{\lambda B(\bar{k})} & A(k) \end{pmatrix} \qquad S_L(k) = \begin{pmatrix} \overline{\mathcal{A}(\bar{k})} & \mathcal{B}(k) \\ \overline{\lambda \mathcal{B}(\bar{k})} & \mathcal{A}(k) \end{pmatrix}.
\tag{1.7}
$$

Regarding step 2 we note that equations (1.6), (1.7) motivate the following definitions: let the vectors

$$
(\phi_1(x,k),\phi_2(x,k))^\dagger \qquad (\Phi_1(t,k),\Phi_2(t,k))^\dagger \qquad (\varphi_1(t,k),\varphi_2(t,k))^\dagger \tag{1.8a}
$$

solve the x-part of the Lax pair evaluated at $t=0$, the t-part of the Lax pair evaluated at $x=0$ and the t-part of the Lax pair evaluated at $x=L$, respectively, and let these vectors satisfy the boundary conditions

$$
\begin{aligned}
&(\phi_1(L,k),\phi_2(L,k))^\dagger = (0,1)^\dagger\\
&(\Phi_1(0,k),\Phi_2(0,k))^\dagger = (0,1)^\dagger\\
&(\varphi_1(0,k),\varphi_2(0,k))^\dagger = (0,1)^\dagger.
\end{aligned}
\tag{1.8b}
$$

Define the spectral functions (1.2) by

$$
\begin{aligned}
&a(k) = \phi_2(0,k) \qquad b(k) = \phi_1(0,k)\\
&A(k) = A(T,k) \qquad B(k) = B(T,k) \qquad \mathcal{A}(k) = \mathcal{A}(T,k) \qquad \mathcal{B}(k) = \mathcal{B}(T,k)
\end{aligned}
$$

where

$$
\begin{aligned}
&A(t,k) = \overline{\Phi_2(t,\bar{k})} \qquad B(t,k) = -\mathrm{e}^{4\mathrm{i}k^2t}\Phi_1(t,k)\\
&\mathcal{A}(t,k) = \overline{\varphi_2(t,\bar{k})} \qquad \mathcal{B}(t,k) = -\mathrm{e}^{4\mathrm{i}k^2t}\varphi_1(t,k).
\end{aligned}
\tag{1.9}
$$

We note that the functions (1.2) depend on the functions (1.3).

The global existence of $q(x,t)$ is based on the unique solvability of the associated RH problem, which in turn is based on the distinctive nature of the functions defining the jump matrices: these functions have explicit (x,t) dependence in an exponential form and they involve the spectral functions $s(k)$, $S(k)$, $S_L(k)$, which have the symmetry properties expressed in equations (1.7). Using these facts it can be shown that the associated homogeneous RH problem has only the trivial solution (i.e. there exists a vanishing lemma). The proof that $q(x,t)$ solves the given nonlinear PDE uses the standard arguments of the dressing method [2]. The proof that $q(0,t) = q_0(x)$, is based on the fact that the RH problem satisfied at $t = 0$ is equivalent to a RH problem defined in terms of $s(k)$ which characterizes $q_0(x)$. The proofs that $\left\{\partial_x^l q(0,t) = g_l(t)\right\}_0^1$ and that $\left\{\partial_x^l q(L,t) = f_l(t)\right\}_0^1$, make crucial use of the global relation (1.4). Indeed, it can be shown that the RH problems at $x = 0$ and at $x = L$, are equivalent to RH problems involving only $S(k)$ and $S_L(k)$ (which in turn characterize $\{g_l(t)\}_{l=0}^1$ and $\{f_l(t)\}_{l=0}^1$), if and only if the spectral functions satisfy the global relation. Thus this relation is not only a necessary but also a *sufficient* condition for existence. Hence given $\{q_0, g_0, f_0\}$, the main problem becomes to show that the global relation characterizes g_1 and f_1.

The analysis of step 3 is based on the Gelfand–Levitan–Marchenko representation of the eigenfunctions $\Phi = (\Phi_1, \Phi_2)^\dagger$ and $\varphi = (\varphi_1, \varphi_2)^\dagger$. Using these representation, it can be shown [3] that Φ can be expressed in terms of four functions $\{M_j(t,s), L_j(t,s)\}_1^2, -t < s < t, t > 0$, satisfying four PDEs (see [3]) as well as the boundary conditions

$$L_1(t,t) = \frac{\mathrm{i}}{2} g_1(t) \qquad L_2(t,-t) = 0 \qquad M_1(t,t) = g_0(t) \qquad M_2(t,-t) = 0. \tag{1.10}$$

Similarly φ can be expressed in terms of the four functions $\{\mathcal{M}_j(t,s), \mathcal{L}_j(t,s)\}_1^2$ satisfying

$$\mathcal{L}_1(t,t) = \frac{\mathrm{i}}{2} f_1(t) \qquad \mathcal{L}_2(t,-t) = 0 \qquad \mathcal{M}_1(t,t) = f_0(t) \qquad \mathcal{M}_2(t,-t) = 0. \tag{1.11}$$

Using definitions (1.9) it can be shown that [4]

$$\begin{aligned} A(t,k) &= 1 + \int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} [2\overline{L_2}(t, t-2\tau) - \mathrm{i}\lambda g_0(t)\overline{M_1}(t, t-2\tau) + 2k\overline{M_2}(t, t-2\tau)]\,\mathrm{d}\tau, \\ B(t,k) &= -\int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} [2L_1(t, 2\tau - t) - \mathrm{i}g_0(t)M_2(t, 2\tau - t) + 2kM_1(t, 2\tau - t)]\,\mathrm{d}\tau. \end{aligned} \tag{1.12}$$

Similar expressions are valid for $\mathcal{A}$ and $\mathcal{B}$, where L_1, M_1, M_2, g_0 are replaced by $\mathcal{L}_1, \mathcal{M}_1, \mathcal{M}_2, f_0$ respectively. Substituting the expressions for $A, B, \mathcal{A}, \mathcal{B}$ in the global relation (1.4) and letting $k \to -k$ in the resulting equation, we obtain two relations coupling

$$g_0, f_0, L_1, M_1, M_2, \mathcal{L}_1, \mathcal{M}_1, \mathcal{M}_2. \tag{1.13}$$

It is remarkable that these two relations can be explicitly solved for g_1 and f_1 in terms of the quantities appearing in (1.13).

Having solved the global relation it is now more convenient to formulate the final result in terms of the functions

$$\{\hat{L}_j(t,k), \hat{M}_j(t,k), \hat{\mathcal{L}}_j(t,k), \hat{\mathcal{M}}_j(t,k)\}_{j=1}^2 \tag{1.14}$$

where

$$\hat{L}_j(t,k) = \int_{-t}^t \mathrm{e}^{2\mathrm{i}k^2(s-t)} L_j(t,s)\,\mathrm{d}s \tag{1.15}$$

and similarly for $\hat{M}_j, \hat{\mathcal{L}}_j, \hat{\mathcal{M}}_j$. Using this notation, the explicit formulae of g_1 and f_1 in terms of the quantities appearing in (1.13) can be expressed explicitly in terms of

$$\{g_0, f_0, \hat{L}_1, \hat{M}_1, \hat{M}_2, \hat{\mathcal{L}}_1, \hat{\mathcal{M}}_1, \hat{\mathcal{M}}_2\}$$

see equations (4.7) and (4.8). The Gelfand–Levitan–Marchenko representations imply that $\{\hat{L}_j, \hat{M}_j\}_{j=1}^2$, for $t > 0$, $k \in \mathbb{C}$, satisfy the ODEs

$$\begin{aligned}
&\hat{L}_{1_t} + 4\mathrm{i}k^2\hat{L}_1 = \mathrm{i}g_1(t)\hat{L}_2 + \chi_1(t)\hat{M}_1 + \chi_2(t)\hat{M}_2 + \mathrm{i}g_1(t)\\
&\hat{L}_{2_t} = -\mathrm{i}\lambda\overline{g_1}(t)\hat{L}_1 - \chi_1(t)\hat{M}_2 + \lambda\bar{\chi}_2(t)\hat{M}_1\\
&\hat{M}_{1_t} + 4\mathrm{i}k^2\hat{M}_1 = 2g_0(t)\hat{L}_2 + \mathrm{i}g_1(t)\hat{M}_2 + 2g_0(t)\\
&\hat{M}_{2_t} = 2\lambda\overline{g_0}(t)\hat{L}_1 - \mathrm{i}\lambda\overline{g_1}(t)\hat{M}_1
\end{aligned} \tag{1.16}$$

as well as the initial conditions

$$\hat{L}_j(0,k) = \hat{M}_j(0,k) = 0 \qquad j = 1, 2$$

where the functions $\chi_1(t)$ and $\chi_2(t)$ are defined by

$$\chi_1(t) = \frac{\lambda}{2}(g_0\overline{g_1} - \overline{g_0}g_1) \qquad \chi_2(t) = \frac{1}{2}\frac{\mathrm{d}g_0}{\mathrm{d}t} - \rho|g_0|^2g_0. \tag{1.17}$$

$\{\hat{\mathcal{L}}_j, \hat{\mathcal{M}}_j\}_{j=1}^2$ satisfy similar equations with g_0, g_1 replaced by f_0, f_1.

Substituting the expressions for g_1 and f_1 from (4.7), (4.8) in equations (1.16) and the analogous equations for $\{\hat{\mathcal{L}}_j, \hat{\mathcal{M}}_j\}_{j=1}^2$, we obtain a system of nonlinear Volterra integral equations for the functions $\{\hat{L}_j, \hat{M}_j, \hat{\mathcal{L}}_j, \hat{\mathcal{M}}_j\}_{j=1}^2$ in terms of g_0 and f_0. It is shown in [20] that equations (1.16) imply

$$|k\hat{M}_2|^2 - \rho|k\hat{M}_1|^2 + \left|1 + \hat{L}_2 + \frac{\mathrm{i}\rho}{2}\bar{g}_0(t)\hat{M}_1\right|^2 - \rho\left|\hat{L}_1 - \frac{\mathrm{i}}{2}g_0(t)\hat{M}_2{}^2\right| = 1 \qquad \operatorname{Im} k^2 = 0.$$

This equation implies that for the focusing case ($\rho = -1$) there exists a global solution. The question of existence for the defocusing case ($\rho = 1$) remains open.

Organization of the paper and notations. Steps 1–3 are implemented in sections 2–4.

In addition to the notation (1.7) the following notation will also be used:

$$s(k)\,\mathrm{e}^{\mathrm{i}kL\hat{\sigma}_3}S_L(k) \equiv s(k)\,\mathrm{e}^{\mathrm{i}kL\sigma_3}S_L(k)\,\mathrm{e}^{-\mathrm{i}kL\sigma_3} = \begin{pmatrix} \overline{\alpha(\bar{k})} & \beta(k) \\ \lambda\overline{\beta(\bar{k})} & \alpha(k) \end{pmatrix}. \tag{1.18}$$

$\mu^{(*)}$ denotes a function which is analytic and bounded for $\{k \in \mathbb{C}, \arg k \in L_*\}$, where

$$\begin{aligned}
&L_1 : \left[0, \frac{\pi}{2}\right] \qquad L_2 : \left[\frac{\pi}{2}, \pi\right] \qquad L_3 : \left[\pi, \frac{3\pi}{2}\right]\\
&L_4 : \left[\frac{3\pi}{2}, \pi\right] \qquad L_{12} : L_1 \cup L_2 \qquad \text{etc.}
\end{aligned} \tag{1.19}$$

2. A Riemann–Hilbert formulation under the assumption of existence

The NLS equation admits the Lax pair [17] formulation [18]

$$\mu_x + \mathrm{i}k\hat{\sigma}_3\mu = Q\mu \qquad \mu_t + 2\mathrm{i}k^2\hat{\sigma}_3\mu = \tilde{Q}\mu \tag{2.1}$$

where $\mu(x,t,k)$ is a 2×2 matrix-valued function, $\hat{\sigma}_3$ is defined by

$$\hat{\sigma}_3 \cdot = [\sigma_3, \cdot] \qquad \sigma_3 = \mathrm{diag}(1, -1) \tag{2.2}$$

and the 2×2 matrices Q, $\tilde{Q}$ are defined by

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \lambda\bar{q}(x,t) & 0 \end{pmatrix} \qquad \tilde{Q}(x,t,k) = 2kQ - \mathrm{i}Q_x\sigma_3 - \mathrm{i}\lambda|q|^2\sigma_3 \qquad \lambda = \pm 1. \tag{2.3}$$

The definition of $\hat{\sigma}_3$ implies that if A is a 2×2 matrix, then

$$\mathrm{e}^{\hat{\sigma}_3} A = \mathrm{e}^{\sigma_3} A\, \mathrm{e}^{-\sigma_3}. \tag{2.4}$$

Equations (2.1) can be rewritten as

$$d\big(\mathrm{e}^{\mathrm{i}(kx+2k^2t)\hat{\sigma}_3}\mu(x,t,k)\big) = W(x,t,k) \tag{2.5}$$

where the closed 1-form W is defined by

$$W = \mathrm{e}^{\mathrm{i}(kx+2k^2t)\hat{\sigma}_3}(Q\mu\,\mathrm{d}x + \tilde{Q}\mu\,\mathrm{d}t). \tag{2.6}$$

Throughout this section we *assume* that there exists a sufficiently smooth solution $q(x,t)$, $x \in [0,L]$, $t \in [0,T]$, of the NLS equation.

A solution of equation (2.5) is given by

$$\mu_*(x,t,k) = I + \int_{(x_*,t_*)}^{(x,t)} \mathrm{e}^{-\mathrm{i}(kx+2k^2t)\hat{\sigma}_3} W(y,\tau,k) \tag{2.7}$$

where (x_*,t_*) is an arbitrary point in the domain $x \in [0,L]$, $t \in [0,T]$, and the integral denotes a line integral connecting smoothly the points indicated. Following [1] we choose the point (x_*,t_*) as each of the corners of the polygonal domain. Thus we define four different solutions $\mu_1,\ldots,\mu_4$, corresponding to $(0,T)$, $(0,0)$, $(L,0)$, (L,T), see figure 1.1.

By splitting the line integrals into integrals parallel to the t and the x axis we find

$$\mu_2(x,t,k) = I + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_2)(y,t,k)\,\mathrm{d}y + \mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}\int_0^t \mathrm{e}^{-2\mathrm{i}k^2(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_2)(0,\tau,k)\,\mathrm{d}\tau \tag{2.8}$$

$$\begin{aligned}\mu_3(x,t,k) = I - \int_x^L \mathrm{e}^{-\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_3)(y,t,k)\,\mathrm{d}y + \mathrm{e}^{-\mathrm{i}k(x-L)\hat{\sigma}_3}\\ \times \int_0^t \mathrm{e}^{-2\mathrm{i}k^2(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_3)(L,\tau,k)\,\mathrm{d}\tau.\end{aligned} \tag{2.9}$$

μ_1 and μ_4 satisfy equations similar to those of μ_2 and μ_3 where $\int_0^t$ is replaced by $-\int_t^T$.

Note that all the μ_j are entire functions of *k*.

2.1. Eigenfunctions and their relations

The definitions of μ_j, $j = 1,\ldots,4$, and the notation (1.19) imply

$$\mu_1 = \big(\mu_1^{(2)},\mu_1^{(3)}\big) \qquad \mu_2 = \big(\mu_2^{(1)},\mu_2^{(4)}\big) \qquad \mu_3 = \big(\mu_3^{(3)},\mu_3^{(2)}\big) \qquad \mu_4 = \big(\mu_4^{(4)},\mu_4^{(1)}\big). \tag{2.10}$$

The functions $\mu_1(0,t,k)$, $\mu_2(0,t,k)$, $\mu_3(x,0,k)$, $\mu_3(L,t,k)$, $\mu_4(L,t,k)$ are bounded in larger domains:

$$\begin{aligned}&\mu_1(0,t,k) = \big(\mu_1^{(24)}(0,t,k),\mu_1^{(13)}(0,t,k)\big) \qquad \mu_2(0,t,k) = \big(\mu_2^{(13)}(0,t,k),\mu_2^{(24)}(0,t,k)\big)\\ &\mu_3(x,0,k) = \big(\mu_3^{(34)}(x,0,k),\mu_3^{(12)}(x,0,k)\big) \qquad \mu_3(L,t,k) = \big(\mu_3^{(13)}(L,t,k),\mu_3^{(24)}(L,t,k)\big)\\ &\mu_4(L,t,k) = \big(\mu_4^{(24)}(L,t,k),\mu_4^{(13)}(L,t,k)\big).\end{aligned} \tag{2.11}$$

The matrices Q and $\tilde{Q}$ are traceless, thus

$$\det\mu_j(x,t,k) = 1 \qquad j = 1,\ldots,4. \tag{2.12}$$

The definitions of $\mu_j^{(*)}$ imply that in the domains where these functions are bounded, they satisfy

$$\mu_j^{(*)}(x,t,k) = I_j^{(*)} + O\left(\frac{1}{k}\right) \qquad k \to \infty \tag{2.13}$$

where vector $I_j^{(*)}$ is either $(0,1)^\dagger$ or $(1,0)^\dagger$, depending on which column of μ_j is denoted by $\mu_j^{(*)}$.

The functions μ_j are related by the equations

$$\mu_3(x,t,k) = \mu_2(x,t,k)\,\mathrm{e}^{-\mathrm{i}(kx+2k^2t)\hat{\sigma}_3} s(k) \tag{2.14}$$

$$\mu_1(x,t,k) = \mu_2(x,t,k)\,\mathrm{e}^{-\mathrm{i}(kx+2k^2t)\hat{\sigma}_3} S(k) \tag{2.15}$$

$$\mu_4(x,t,k) = \mu_3(x,t,k)\,\mathrm{e}^{-\mathrm{i}[k(x-L)+2\mathrm{i}k^2t]\hat{\sigma}_3} S_L(k). \tag{2.16}$$

Evaluating equation (2.14) at $x = t = 0$ we find $s(k) = \mu_3(0,0,k)$. Evaluating equation (2.15) at $x = t = 0$ we find $S(k) = \mu_1(0,0,k)$; evaluating equation (2.15) at $x = 0, t = T$ we find $S(k) = \left(\mathrm{e}^{2\mathrm{i}k^2T\hat{\sigma}_3}\mu_2(0,T,k)\right)^{-1}$. Evaluating equation (2.15) at $x = L, t = 0$ we find $S_L(k) = \mu_4(L,0,k)$; evaluating equation (2.16) at $x = L, t = T$ we find $S_L(k) = \left(\mathrm{e}^{2\mathrm{i}k^2T\hat{\sigma}_3}\mu_3(L,T,k)\right)^{-1}$. Equations (2.14) and (2.16) imply

$$\mu_4(x,t,k) = \mu_2(x,t,k)\,\mathrm{e}^{-\mathrm{i}(kx+2k^2t)\hat{\sigma}_3}\left(s(k)\,\mathrm{e}^{\mathrm{i}kL\hat{\sigma}_3} S_L(k)\right). \tag{2.17}$$

The symmetry properties of Q and $\tilde{Q}$ imply

$$(\mu(x,t,k))_{11} = \overline{(\mu(x,t,\bar{k}))_{22}} \qquad (\mu(x,t,k))_{21} = \overline{\lambda(\mu(x,t,\bar{k}))_{12}}. \tag{2.18}$$

The definitions of $\mu_3(0,0,k)$, $\mu_2(0,T,k)$, $\mu_3(L,0,k)$ imply

$$s(k) = I - \int_0^L \mathrm{e}^{\mathrm{i}ky\hat{\sigma}_3}(Q\mu_3)(y,0,k)\,\mathrm{d}y \tag{2.19}$$

$$S^{-1}(k) = I + \int_0^T \mathrm{e}^{2\mathrm{i}k^2\tau\hat{\sigma}_3}(\tilde{Q}\mu_2)(0,\tau,k)\,\mathrm{d}\tau \tag{2.20}$$

$$S_L^{-1}(k) = I + \int_0^T \mathrm{e}^{2\mathrm{i}k^2\tau\hat{\sigma}_3}(\tilde{Q}\mu_3)(L,\tau,k)\,\mathrm{d}\tau. \tag{2.21}$$

The symmetry conditions (2.18) justify the notation (1.7).

Equations (2.11), the determinant condition (2.12) and the large k behaviour of μ_j imply the following properties:

$\underline{a(k), b(k)}$

- $a(k), b(k)$ are entire functions.
- $a(k)\overline{a(\bar{k})} - \lambda b(k)\overline{b(\bar{k})} = 1, k \in \mathbb{C}$.
- $$a(k) = 1 + O\left(\frac{1+\mathrm{e}^{2\mathrm{i}kL}}{k}\right) \qquad b(k) = O\left(\frac{1+\mathrm{e}^{2\mathrm{i}kL}}{k}\right) \qquad k \to \infty.$$

In particular,

$$a(k) \quad b(k) \quad \overline{a(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL} \quad \overline{b(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL} \quad \text{are bounded for} \quad \arg k \in [0,\pi]. \tag{2.22}$$

$\underline{A(k), B(k)}$

- $A(k)$, $B(k)$ are entire functions.
- $A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1$, $k \in \mathbb{C}$.
- $$A(k) = 1 + O\left(\frac{1 + \mathrm{e}^{4\mathrm{i}k^2 T}}{k}\right) \qquad B(k) = O\left(\frac{1 + \mathrm{e}^{4\mathrm{i}k^2 T}}{k}\right) \qquad k \to \infty. \tag{2.23}$$

  In particular

  $$A(k) \quad B(k) \quad \text{are bounded for} \quad \arg k \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right].$$

$\underline{\mathcal{A}(k), \mathcal{B}(k)}$

Same as $A(k)$, $B(k)$.

2.2. The global relation

Proposition 2.1. *Let the spectral functions $a(k)$, $b(k)$, $A(k)$, $B(k)$, $\mathcal{A}(k)$, $\mathcal{B}(k)$ be defined in equations (1.7), where $s(k)$, $S(k)$, $S_L(k)$ are defined by equations (1.6), and μ_2, μ_3 are defined by equations (2.8), (2.9) in terms of the smooth function $q(x, t)$. These spectral functions are not independent but they satisfy the global relation (1.4) where $c^+(k)$ denotes the (12) element of $-\int_0^L [\exp(\mathrm{i}ky\hat{\sigma}_3)](Q\mu_4)(y, T, k)\,\mathrm{d}y$, and μ_4 is defined by an equation similar to μ_3 with $\int_0^t$ replaced by $-\int_t^T$.*

Proof. Evaluating equation (2.17) at $x = 0, t = T$ and writing $\mu_2(0, T, k)$ in terms of $S(k)$ we find

$$\mu_4(0, T, k) = \mathrm{e}^{-2\mathrm{i}k^2 T\hat{\sigma}_3}(S^{-1}(k)s(k)\,\mathrm{e}^{\mathrm{i}kL\hat{\sigma}_3}S_L(k)).$$

Multiplying this equation by $\exp[2\mathrm{i}k^2 T\hat{\sigma}_3]$ and using the definition of $\mu_4(x, T, k)$ we find

$$-I + S^{-1}s\left(\mathrm{e}^{\mathrm{i}kL\hat{\sigma}_3}S_L\right) + \mathrm{e}^{2\mathrm{i}k^2 T\hat{\sigma}_3}\int_0^L \mathrm{e}^{\mathrm{i}ky\hat{\sigma}_3}(Q\mu_4)(y, T, k)\,\mathrm{d}y = 0.$$

The (12) element of this equation is equation (1.4). □

2.3. The jump conditions

Let $M(x, t, k)$ be defined by

$$\begin{aligned}
M_+ &= \left(\frac{\mu_2^{(1)}}{\alpha(k)}, \mu_4^{(1)}\right) \quad \arg k \in \left[0, \frac{\pi}{2}\right] \qquad M_- = \left(\frac{\mu_1^{(2)}}{d(k)}, \mu_3^{(2)}\right) \quad \arg k \in \left[\frac{\pi}{2}, \pi\right] \\
M_+ &= \left(\mu_3^{(3)}, \frac{\mu_1^{(3)}}{\overline{\overline{d(\bar{k})}}}\right) \quad \arg k \in \left[\pi, \frac{3\pi}{2}\right] \qquad M_- = \left(\mu_4^{(4)}, \frac{\mu_2^{(4)}}{\overline{\overline{\alpha(\bar{k})}}}\right) \quad \arg k \in \left[\frac{3\pi}{2}, 2\pi\right]
\end{aligned} \tag{2.24}$$

where the scalars $d(k)$ and $\alpha(k)$ are defined below, see (2.30) and (2.31).

These definitions imply

$$\det M(x, t, k) = 1 \tag{2.25}$$

and

$$M(x, t, k) = I + O\left(\frac{1}{k}\right) \qquad k \to \infty. \tag{2.26}$$

Proposition 2.2. *Let $M(x,t,k)$ be defined by equations (2.24), where $\mu_2(x,t,k)$, $\mu_3(x,t,k)$ are defined by equations (2.8), (2.9), $\mu_1(x,t,k)$, $\mu_4(x,t,k)$ are defined by similar equations with $\int_0^t$ replaced by $-\int_t^T$ and $q(x,t)$ is a smooth function. Then M satisfies the 'jump' condition*

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k) \qquad k \in \mathbb{R} \cup \mathrm{i}\mathbb{R} \tag{2.27}$$

where the 2×2 matrix J is defined by

$$J = \begin{cases} J_2 & \arg k = 0 \\ J_1 & \arg k = \frac{\pi}{2} \\ J_4 \equiv J_3 J_2^{-1} J_1 & \arg k = \pi \\ J_3 & \arg k = \frac{3\pi}{2} \end{cases} \tag{2.28}$$

and

$$J_1 = \begin{pmatrix} \frac{\delta(k)}{d(k)} & -\mathcal{B}(k)\,\mathrm{e}^{2\mathrm{i}kL}\,\mathrm{e}^{-2\mathrm{i}\theta} \\ \frac{\lambda\overline{\mathcal{B}(\bar{k})}}{d(k)\alpha(k)}\,\mathrm{e}^{2\mathrm{i}\theta} & \frac{a(k)}{\alpha(k)} \end{pmatrix} \qquad J_3 = \begin{pmatrix} \frac{\overline{\delta(\bar{k})}}{\overline{d(\bar{k})}} & \frac{-B(k)}{\overline{d(\bar{k})\alpha(\bar{k})}}\,\mathrm{e}^{-2\mathrm{i}\theta} \\ \lambda\overline{\mathcal{B}(\bar{k})}\,\mathrm{e}^{-2\mathrm{i}kL}\,\mathrm{e}^{2\mathrm{i}\theta} & \frac{\overline{a(\bar{k})}}{\overline{\alpha(\bar{k})}} \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 1 & -\frac{\beta(k)}{\overline{\alpha(\bar{k})}}\,\mathrm{e}^{-2\mathrm{i}\theta} \\ \lambda\frac{\overline{\beta(\bar{k})}}{\alpha(k)}\,\mathrm{e}^{2\mathrm{i}\theta} & \frac{1}{|\alpha(k)|^2} \end{pmatrix} \qquad \theta(x,t,k) = kx + 2k^2 t. \tag{2.29}$$

$$\alpha(k) = a(k)\mathcal{A}(k) + \lambda\overline{b(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL}\mathcal{B}(k) \qquad \beta(k) = b(k)\mathcal{A}(k) + \overline{a(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL}\mathcal{B}(k) \tag{2.30}$$

$$d(k) = a(k)\overline{A(\bar{k})} - \lambda b(k)\overline{B(\bar{k})} \qquad \delta(k) = \alpha(k)\overline{A(\bar{k})} - \lambda\beta(k)\overline{B(\bar{k})}. \tag{2.31}$$

Proof. Writing equations (2.17), (2.14) and (2.15) in vector form we find

$$\mu_4^{(4)} = \bar{\alpha}\mu_2^{(1)} + \lambda\bar{\beta}e\mu_2^{(4)} \qquad \mu_4^{(1)} = \beta\bar{e}\mu_2^{(1)} + \alpha\mu_2^{(4)} \tag{2.32}$$

$$\mu_3^{(3)} = \bar{a}\mu_2^{(1)} + \lambda\bar{b}e\mu_2^{(4)} \qquad \mu_3^{(2)} = b\bar{e}\mu_2^{(1)} + a\mu_2^{(4)} \tag{2.33}$$

$$\mu_1^{(2)} = \bar{A}\mu_2^{(1)} + \lambda\bar{B}e\mu_2^{(4)} \qquad \mu_1^{(3)} = \bar{B}\bar{e}\mu_2^{(1)} + A\mu_2^{(4)} \tag{2.34}$$

where $e = \exp(2\mathrm{i}\theta)$. Recall that α and β are the (22) and (12) elements of $s\,\mathrm{e}^{\mathrm{i}kL\hat{\sigma}_3}S_L$ (see (1.18)), thus

$$\alpha(k)\overline{\alpha(\bar{k})} - \lambda\beta(k)\overline{\beta(\bar{k})} = 1. \tag{2.35}$$

Rearranging equations (2.32) and using equation (2.35) we find the jump condition across $\arg k = 0$.

In order to derive the jump condition across $\arg k = \frac{\pi}{2}$, we first eliminate $\mu_2^{(4)}$ from equations (2.32*b*) and (2.34*a*):

$$\mu_1^{(2)} = \frac{\delta\mu_2^{(1)}}{\alpha} + \frac{\lambda\bar{B}\,\mathrm{e}}{\alpha}\mu_4^{(1)}. \tag{2.36a}$$

We then eliminate $\mu_2^{(4)}$ from equations (2.32*b*) and (2.33*b*):

$$\mu_3^{(2)} = (b\alpha - a\beta)\frac{\bar{e}\mu_2^{(1)}}{\alpha} + \frac{a\mu_4^{(1)}}{\alpha}. \tag{2.36b}$$

Using the identity

$$a\beta - b\alpha = \mathrm{e}^{2\mathrm{i}kL}\mathcal{B} \tag{2.37}$$

and dividing equation (2.36*a*) by d, equations (2.36) define the jump across $\arg k = \frac{\pi}{2}$.

The jump across $\arg k = \frac{3\pi}{2}$ follows from symmetry considerations and the jump across $\arg k = \pi$ follows from the fact that the product of the jump matrices must equal the identity.

We note that the jump matrices have unit determinant; in particular regarding J_1 we note that

$$d\alpha - \lambda \bar{B}\mathcal{B}\,\mathrm{e}^{2\mathrm{i}kL} = a\delta. \tag{2.38}$$

Indeed, the lhs of this equation equals

$$\alpha(a\bar{A} - \lambda b\bar{B}) - \lambda\bar{B}\mathcal{B}\,\mathrm{e}^{2\mathrm{i}kL} = \alpha a\bar{A} - \lambda\bar{B}(\mathcal{B}\,\mathrm{e}^{2\mathrm{i}kL} + \alpha b) = \alpha a\bar{A} - \lambda\bar{B}a\beta = a\delta$$

where we have used the identity (2.37). □

2.4. The residue relations

Proposition 2.3. *Let $\alpha(k)$ and $d(k)$ be defined by equations (2.30) and (2.31) in terms of the spectral functions considered in proposition 2.1. Assume that*

- *$\alpha(k)$ has simple zeros, $\{\nu_j\}$, $\arg\nu_j \in \left(0, \frac{\pi}{2}\right)$, and has* no *zeros for $\arg k = 0$ and $\arg k = \frac{\pi}{2}$.*
- *$d(k)$ has simple zeros, $\{\lambda_j\}$, $\arg\lambda_j \in \left(\frac{\pi}{2}, \pi\right)$, and has* no *zeros for $\arg k = \frac{\pi}{2}$ and $\arg k = \pi$.*
- *None of the zeros of $d(k)$ for $\arg k \in \left(\frac{\pi}{2}, \pi\right)$ coincides with any of the zeros of $a(k)$.*
- *None of the zeros of $\alpha(k)$ for $\arg k \in \left(0, \frac{\pi}{2}\right)$ coincides with any of the zeros of $a(k)$.*

Let $[M]_1$ and $[M]_2$ denote the first and the second column of the matrix M. Then

$$\mathrm{Res}_{k=\nu_j}[M(x,t,k)]_1 = c_j^{(1)}\,\mathrm{e}^{4\mathrm{i}\nu_j^2 t + 2\mathrm{i}\nu_j x}[M(x,t,\nu_j)]_2 \tag{2.39}$$

$$\mathrm{Res}_{k=\bar{\nu}_j}[M(x,t,k)]_2 = \lambda\overline{c_j^{(1)}}\,e^{-4\mathrm{i}\bar{\nu}_j^2 t - 2\mathrm{i}\bar{\nu}_j x}[M(x,t,\bar{\nu}_j)_1 \tag{2.40}$$

$$\mathrm{Res}_{k=\lambda_j}[M(x,t,k)]_1 = c_j^{(2)}\,\mathrm{e}^{4\mathrm{i}\lambda_j^2 t + 2\mathrm{i}\lambda_j x}[M(x,t,\lambda_j)]_2 \tag{2.41}$$

$$\mathrm{Res}_{k=\bar{\lambda}_j}[M(x,t,k)]_2 = \lambda\overline{c_j^{(2)}}\,\mathrm{e}^{-4\mathrm{i}\bar{\lambda}_j^2 t - 2\mathrm{i}\bar{\lambda}_j x}[M(x,t,\bar{\lambda}_j)]_1 \tag{2.42}$$

where

$$c_j^{(1)} = \frac{a(\nu_j)}{\mathrm{e}^{2\mathrm{i}\nu_j L}\mathcal{B}(\nu_j)\dot{\alpha}(\nu_j)} \qquad c_j^{(2)} = \frac{\lambda\overline{B(\bar{\lambda}_j)}}{a(\lambda_j)\dot{d}(\lambda_j)}. \tag{2.43}$$

Proof. Let us first note that the assumptions of the proposition, the definitions of the quantities $\alpha(k)$ and $d(k)$ (see (2.30) and (2.31)) and the identities

$$A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1 \qquad \mathcal{A}(k)\overline{\mathcal{A}(\bar{k})} - \lambda\mathcal{B}(k)\overline{\mathcal{B}(\bar{k})} = 1$$

imply that

$$B(\bar{\lambda}_j) \neq 0 \qquad \text{and} \qquad \mathcal{B}(\nu_j) \neq 0$$

so that the rhs of equations (2.43) are well defined. We proceed now with the proof of the proposition.

The matrix $J_1(x,t,k)$ defined in equations (2.29) can be factorized as follows:

$$J_1(x,t,k) = \begin{pmatrix} \frac{\alpha(k)}{a(k)} & -\mathcal{B}(k)\,\mathrm{e}^{2\mathrm{i}kL}\,\mathrm{e}^{-2\mathrm{i}\theta} \\ 0 & \frac{a(k)}{\alpha(k)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\lambda\overline{B(\bar{k})}\,\mathrm{e}^{2\mathrm{i}\theta}}{a(k)d(k)} & 1 \end{pmatrix}. \tag{2.44}$$

Indeed, the entries (12), (21), (22) are equal identically. The entries (11) are equal iff

$$\delta a = \alpha d - \lambda \mathcal{B}\overline{B(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL}.$$

Replacing δ and d in this equation by their definitions (see equations (2.31)) we find the identity (2.37).

Using the factorization (2.44), the jump condition $M_- = M_+ J_1$ becomes

$$\left(\frac{\mu_1^{(2)}}{d}, \mu_3^{(2)}\right)\begin{pmatrix} 1 & 0 \\ \frac{-\lambda \bar{B}\,\mathrm{e}^{2\mathrm{i}\theta}}{ad} & 1 \end{pmatrix} = \left(\frac{\mu_2^{(1)}}{\alpha}, \mu_4^{(1)}\right)\begin{pmatrix} \frac{\alpha}{a} & -\mathcal{B}\,\mathrm{e}^{2\mathrm{i}kL}\,\mathrm{e}^{-2\mathrm{i}\theta} \\ 0 & \frac{a}{\alpha} \end{pmatrix}. \tag{2.45}$$

Evaluating the second column of this equation at $k = \nu_j$ (we recall that all the functions involved are entire) we find

$$0 = -\mathcal{B}(\nu_j)\,\mathrm{e}^{2\mathrm{i}\nu_j L}\,\mathrm{e}^{-2\mathrm{i}\theta(\nu_j)}\mu_2^{(1)}(\nu_j) + a(\nu_j)\mu_4^{(1)}(\nu_j) \tag{2.46}$$

where for convenience of notation we have suppressed the x, t dependence of $\mu_2^{(1)}, \mu_4^{(1)}, \theta$. Hence,

$$\mathrm{Res}_{k=\nu_j}[M(x,t,k)]_1 = \frac{\mu_2^{(1)}(x,t,\nu_j)}{\dot{\alpha}(\nu_j)} = \frac{a(\nu_j)\,\mathrm{e}^{2\mathrm{i}\theta(x,t,\nu_j)}\mu_4^{(1)}(x,t,\nu_j)}{\mathrm{e}^{2\mathrm{i}\nu_j L}\mathcal{B}(\nu_j)\dot{\alpha}(\nu_j)}$$

which, using $\mu_4^{(1)}(x,t,\nu_j) = [M(x,t,\nu_j)]_2$, becomes equation (2.39).

Similarly, evaluating the first column of equation (2.45) at $k = \lambda_j$ we find

$$0 = \mu_1^{(2)}(\lambda_j) - \frac{\lambda\overline{B(\bar{\lambda}_j)}\,\mathrm{e}^{2\mathrm{i}\theta(\lambda_j)}}{a(\lambda_j)}\mu_3^{(2)}(\lambda_j).$$

Hence

$$\mathrm{Res}_{k=\lambda_j}[M(x,t,k)]_1 = \frac{\mu_1^{(2)}(x,t,\lambda_j)}{\dot{d}(\lambda_j)} = \frac{\lambda\overline{B(\bar{\lambda}_j)}\,\mathrm{e}^{2\mathrm{i}\theta(x,t,\lambda_j)\mu_3^{(2)}(x,t,\lambda_j)}}{a(\lambda_j)\dot{d}(\lambda_j)}$$

which yields (2.41).

Equations (2.40) and (2.42) follow from equations (2.39) and (2.41) using symmetry considerations. □

3. Existence under the assumption that the global relation is valid

3.1. The spectral functions

The analysis of section 2 motivates the following definitions and results for the spectral functions. The relevant rigorous analysis can be found in [6].

Definition 3.1 (the spectral functions $a(k), b(k)$)**.** *Given the smooth function $q_0(x)$, define the vector $\phi(x,k) = (\phi_1, \phi_2)^\dagger$ as the unique solution of*

$$\phi_{1_x} + 2\mathrm{i}k\phi_1 = q_0(x)\phi_2 \qquad \phi_{2_x} = \lambda\bar{q}_0(x)\phi_1 \qquad 0 < x < L \quad k \in \mathbb{C} \quad \phi(L,k) = (0,1)^\dagger. \tag{3.1}$$

Given $\phi(x,k)$ define the functions $a(k)$ and $b(k)$ by

$$a(k) = \phi_2(0,k) \qquad b(k) = \phi_1(0,k) \quad k \in \mathbb{C}. \tag{3.2}$$

Properties of $a(k), b(k)$:

- $a(k), b(k)$ are entire functions.
- $a(k)\overline{a(\bar{k})} - \lambda b(k)\overline{b(\bar{k})} = 1, k \in \mathbb{C}$.

- $$a(k) = 1 + O\left(\frac{1 + \mathrm{e}^{2\mathrm{i}kL}}{k}\right) \qquad b(k) = O\left(\frac{1 + \mathrm{e}^{2\mathrm{i}kL}}{k}\right) \qquad k \to \infty.$$

  In particular,

  $$a(k) \quad b(k) \quad \overline{a(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL} \quad \overline{b(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL} \quad \text{are bounded for} \quad \arg k \in [0, \pi]. \tag{2.22}$$

We shall also assume that $a(k)$ has at most simple zeros, $\{k_j\}$, for $\mathrm{Im}\, k_j > 0$ and has no zeros for $\mathrm{Im}\, k = 0$.

Remark 3.1. Definition 3.1 gives rise to the map

$$\mathbb{S} : \{q_0(x)\} \to \{a(k), b(k)\}. \tag{3.3a}$$

The inverse of this map,

$$\mathbb{Q} : \{a(k), b(k)\} \to \{q_0(x)\} \tag{3.3b}$$

can be defined as follows:

$$q_0(x) = 2\mathrm{i} \lim_{k\to\infty} (kM^{(x)}(x, k))_{12} \tag{3.4}$$

where $M^{(x)}(x, k)$ is the unique solution of the following RH problem:

- $$M^{(x)}(x, k) = \begin{cases} M_-^{(x)}(x, k) & \mathrm{Im}\, k \leqslant 0 \\ M_+^{(x)}(x, k) & \mathrm{Im}\, k \geqslant 0, \end{cases}$$

  is a sectionally meromorphic function with unit determinant.

- $$M_-^{(x)}(x, k) = M_+^{(x)}(x, k) J^{(x)}(x, k) \qquad k \in \mathbb{R}$$

  where

  $$J^{(x)}(x, k) = \begin{pmatrix} 1 & -\frac{b(k)}{\bar{a}(k)}\,\mathrm{e}^{-2\mathrm{i}kx} \\ \frac{\lambda \bar{b}(k)\,\mathrm{e}^{2\mathrm{i}kx}}{a(k)} & \frac{1}{|a|^2} \end{pmatrix}.$$

- $$M^{(x)}(x, k) = I + O\left(\frac{1}{k}\right) \qquad k \to \infty.$$

- The first column of $M_+^{(x)}$ can have simple poles at $k = k_j$, and the second column of $M_-^{(x)}$ can have simple poles at $k = \bar{k}_j$, where $\{k_j\}$ are the simple zeros of $a(k)$, $\mathrm{Im}\, k_j > 0$. The associated residues are given by

  $$\begin{aligned} \mathrm{Res}_{k=k_j}[M^{(x)}(x, k)]_1 &= \frac{\mathrm{e}^{2\mathrm{i}k_j x}}{\dot{a}(k_j) b(k_j)} [M^{(x)}(x, k_j)]_2, \\ \mathrm{Res}_{k=\bar{k}_j}[M^{(x)}(x, k)]_2 &= \frac{\lambda\,\mathrm{e}^{-2\mathrm{i}\bar{k}_j x}}{\overline{\dot{a}(k_j) b(k_j)}} [M^{(x)}(x, \bar{k}_j)]_1. \end{aligned} \tag{3.5}$$

  It can be shown (see, for example, [6]) that

$$\mathbb{S}^{-1} = \mathbb{Q}. \tag{3.3c}$$

Definition 3.2 (the spectral functions $A(k)$, $B(k)$)**.** *Let*

$$Q^{(0)}(t,k) = 2k\begin{pmatrix} 0 & g_0(t) \\ \lambda\bar{g}_0(t) & 0 \end{pmatrix} - \mathrm{i}\begin{pmatrix} 0 & g_1(t) \\ \lambda\bar{g}_1(t) & 0 \end{pmatrix}\sigma_3 - \mathrm{i}\lambda|g_0(t)|^2\sigma_3 \qquad \lambda = \pm 1. \tag{3.6}$$

Given the smooth functions $g_0(t)$, $g_1(t)$, define the vector $\Phi(t,k) = (\Phi_1, \Phi_2)^\dagger$ as the unique solution of

$$\begin{aligned} &\Phi_{1_t} + 4\mathrm{i}k^2\Phi_1 = Q_{11}^{(0)}\Phi_1 + Q_{12}^{(0)}\Phi_2 \\ &\Phi_{2_t} = Q_{21}^{(0)}\Phi_1 + Q_{22}^{(0)}\Phi_2 \qquad 0 < t < T \quad k \in \mathbb{C} \\ &\Phi(0,k) = (0,1)^\dagger. \end{aligned} \tag{3.7}$$

Given $\Phi(t,k)$ define the functions $A(k)$ and $B(k)$ by

$$A(k) = \overline{\Phi_2(T,\bar{k})} \qquad B(k) = -\Phi_1(T,k)\,\mathrm{e}^{4\mathrm{i}k^2T}. \tag{3.8}$$

Properties of $A(k)$, $B(k)$:

- $A(k)$, $B(k)$ are entire functions.
- $A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1$, $k \in \mathbb{C}$.
-
$$A(k) = 1 + O\left(\frac{1+\mathrm{e}^{4\mathrm{i}k^2T}}{k}\right) \qquad B(k) = O\left(\frac{1+\mathrm{e}^{4\mathrm{i}k^2T}}{k}\right) \qquad k \to \infty. \tag{2.23}$$

  In particular,

$$A(k) \quad B(k) \quad \text{are bounded for} \quad \arg k \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right].$$

We shall also assume that $A(k)$ has at most simple zeros, $\{K_j\}$, for arg $K_j \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$ and has no zeros for $\arg k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Remark 3.2. Definition 3.2 gives rise to the map

$$\mathbb{S}^{(0)} : \{g_0(t), g_1(t)\} \to \{A(k), B(k)\}. \tag{3.9a}$$

The inverse of this map

$$\mathbb{Q}^{(0)} : \{A(k), B(k)\} \to \{g_0(t), g_1(t)\} \tag{3.9b}$$

can be defined as follows,

$$\begin{aligned} g_0(t) &= 2\mathrm{i}\lim_{k\to\infty}(kM^{(0)}(t,k))_{12} \\ g_1(t) &= \lim_{k\to\infty}\{4(k^2M^{(0)}(t,k))_{12} + 2\mathrm{i}g_0(t)k(M^{(0)}(t,k) - I)_{22}\} \end{aligned}$$

where $M^{(0)}(t,k)$ is the unique solution of the following RH problem:

-
$$M^{(0)}(t,k) = \begin{bmatrix} M_+^{(0)}(t,k) & \arg k \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right] \\ M_-^{(0)}(t,k) & \arg k \in \left[\frac{\pi}{2}, \pi\right] \cup \left[\frac{3\pi}{2}, 2\pi\right] \end{bmatrix}$$

  is a sectionally meromorphic function with unit determinant.

- $$M_-^{(0)}(t,k) = M_+^{(0)}(t,k)J^{(0)}(t,k) \qquad k \in \mathbb{R} \cup i\mathbb{R}$$

  where

  $$J^{(0)}(t,k) = \begin{pmatrix} 1 & -\frac{B(k)}{\overline{A(\bar{k})}}\,\mathrm{e}^{-4\mathrm{i}k^2t} \\ \frac{\lambda\overline{B(\bar{k})}\,\mathrm{e}^{4\mathrm{i}k^2t}}{A(k)} & \frac{1}{A(k)\overline{A(\bar{k})}}. \end{pmatrix}$$

- $$M^{(0)}(t,k) = I + O\left(\frac{1}{k}\right) \qquad k \to \infty.$$

- The first column of $M_+^{(0)}(t,k)$ can have simple poles at $k = K_j$ and the second column of $M_-^{(0)}(t,k)$ can have simple poles at $k = \bar{K}_j$, where K_j are the simple zeros of $A(k)$, $\arg k \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$. The associated residues are given by

  $$\mathrm{Res}_{k=K_j}[M^{(0)}(t,k)]_1 = \frac{\exp\left[4iK_j^2t\right]}{\dot{A}(K_j)B(K_j)}[M^{(0)}(t,K_j)]_2,$$

  $$\mathrm{Res}_{k=\bar{K}_j}[M^{(0)}(t,k)]_2 = \frac{\lambda\exp\left[-4\mathrm{i}\bar{K}_j^2t\right]}{\overline{\dot{A}(K_j)}\,\overline{B(K_j)}}[M^{(0)}(t,\bar{K}_j)]_1.$$

  It can be shown, see again [6], that

$$(\mathbb{S}^{(0)})^{-1} = \mathbb{Q}^{(0)}. \tag{3.9c}$$

Definition 3.3 (the spectral functions $\mathcal{A}(k), \mathcal{B}(k)$)**.** *Let $Q^{(L)}(t,k)$ be defined by an equation similar to (3.6) with $g_0(t), g_1(t)$ replaced by $f_0(t), f_1(t)$. Given the smooth functions $f_0(t), f_1(t)$ define the vector $\varphi(t,k)$ by equations similar to (3.7) with $Q^{(0)}(t,k)$ replaced by $Q^{(L)}(t,k)$. Given $\varphi(t,k)$ define $\mathcal{A}(k)$ and $\mathcal{B}(k)$ by*

$$\mathcal{A}(k) = \overline{\varphi_2(T,\bar{k})}, \mathcal{B}(k) = -\varphi_1(T,k)\,\mathrm{e}^{4\mathrm{i}k^2T}. \tag{3.10}$$

Properties of $\mathcal{A}(k), \mathcal{B}(k)$:
Identical to those of $A(k)$, $B(k)$. We will denote the zeros of $\mathcal{A}(k)$ by $\mathcal{K}_j$.

Remark 3.3. The maps

$$\mathbb{S}^{(L)} : \{f_0(t), f_1(t)\} \to \{\mathcal{A}(k), \mathcal{B}(k)\} \tag{3.11a}$$

and

$$\mathbb{Q}^{(L)} : \{\mathcal{A}(k), \mathcal{B}(k)\} \to \{f_0(t), f_1(t)\}, \tag{3.11b}$$

are defined exactly as in remark 3.2, where we use the notation

$$M^{(L)}(t,k) \quad J^{(L)}(t,k) \quad \mathcal{K}_j \quad \text{instead of} \quad M^{(0)}(t,k) \quad J^{(0)}(t,k) \quad K_j. \tag{3.12}$$

In analogy with equation (3.9c) we find

$$(\mathbb{S}^{(L)})^{-1} = \mathbb{Q}^{(L)}. \tag{3.11c}$$

Definition 3.4 (an admissible set)**.** *Given the smooth function $q_0(x)$ define $a(k)$, $b(k)$ according to definition 3.1. Suppose that there exist smooth functions $g_0(t), g_1(t), f_0(t), f_1(t)$, such that*

- *The associated* $A(k)$, $B(k)$, $\mathcal{A}(k)$, $\mathcal{B}(k)$, *defined according to definitions 3.2 and 3.3, satisfy the relation*

$$(a\mathcal{A} + \lambda\bar{b}\,\mathrm{e}^{2\mathrm{i}kL}\mathcal{B})B - (b\mathcal{A} + \bar{a}\,\mathrm{e}^{2\mathrm{i}kL}\mathcal{B})A = \mathrm{e}^{4\mathrm{i}k^2t}c^+(k) \qquad k \in \mathbb{C} \tag{3.13}$$

  where $c^+(k)$ *is an entire function, which is bounded for* $\operatorname{Im} k \geqslant 0$ *and* $c^+(k) = O\left(\frac{1+\mathrm{e}^{2\mathrm{i}kL}}{k}\right)$, $k \to \infty$.
-

$$g_0(0) = q_0(0) \qquad g_1(0) = q_0'(0) \qquad f_0(0) = q_0(L) \qquad f_1(0) = q_0'(L). \tag{3.14}$$

  Then we call the functions $g_0(t)$, $g_1(t)$, $f_0(t)$, $f_1(t)$ *an admissible set of functions with respect to* $q_0(x)$.

3.2. The Riemann–Hilbert problem

Theorem 3.1. *Let* $q_0(x)$ *be a smooth function. Suppose that the set of functions* $g_0(t)$, $g_1(t)$, $f_0(t)$, $f_1(t)$, *is admissible with respect to* $q_0(x)$, *see definition 3.4. Define the spectral functions* $a(k), b(k), A(k), B(k), \mathcal{A}(k), \mathcal{B}(k)$ *in terms of* $q_0(x), g_0(t), g_1(t), f_0(t), f_1(t)$, *according to definitions 3.1, 3.2 and 3.3. Assume that*

- $a(k)$ *has at most simple zeros,* $\{k_j\}$, *for* $\operatorname{Im} k_j > 0$ *and has* no *zeros for* $\operatorname{Im} k = 0$.
- $A(k)$ *has at most simple zeros,* $\{K_j\}$, *for* $\arg K_j \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$ *and has* no *zeros for* $\arg k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
- $\mathcal{A}(k)$ *has at most simple zeros,* $\{\mathcal{K}_j\}$, *for* $\arg \mathcal{K}_j \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$ *and has* no *zeros for* $\arg k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
- *The function*

$$d(k) = a(k)\overline{A(\bar{k})} - \lambda b(k)\overline{B(\bar{k})} \tag{3.15}$$

  has at most simple zeros, $\{\lambda_j\}$, *for* $\arg \lambda_j \in \left(\frac{\pi}{2}, \pi\right)$ *and has* no *zeros for* $\arg\, k = \frac{\pi}{2}$ *and* $\arg\, k = \pi$.
- *The function*

$$\alpha(k) = a(k)\mathcal{A}(k) + \lambda\overline{b(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL}\mathcal{B}(k) \tag{3.16}$$

  has at most simple zeros, $\{\nu_j\}$, *for* $\arg\, \nu_j \in \left(0, \frac{\pi}{2}\right)$ *and has* no *zeros for* $\arg\, k = 0$ $\arg\, k = \frac{\pi}{2}$.
- *None of the zeros of* $a(k)$ *for* $\arg\, k \in \left(\frac{\pi}{2}, \pi\right)$ *coincides with a zero of* $d(k)$.
- *None of the zeros of* $a(k)$ *for* $\arg k \in \left(0, \frac{\pi}{2}\right)$ *coincides with a zero of* $\alpha(k)$.
- *None of the zeros of* $\alpha(k)$ *for* $\arg\, k \in \left(0, \frac{\pi}{2}\right)$ *coincides with a zero of* $A(k)$ *or a zero of* $\mathcal{A}(k)$.
- *None of the zeros of* $d(k)$ *for* $\arg k \in \left(\frac{\pi}{2}, \pi\right)$ *coincides with a zero of* $\overline{A(\bar{k})}$ *or a zero of* $\overline{\mathcal{A}(\bar{k})}$.

Define $M(x, t, k)$ *as the solution of the following* 2×2 *matrix RH problem:*

- M *is sectionally meromorphic in* $\mathbb{C}/\{\mathbb{R} \cup \mathrm{i}\mathbb{R}\}$, *and has unit determinant.*
-

$$M_-(x, t, k) = M_+(x, t, k)J(x, t, k) \qquad k \in \mathbb{R} \cup \mathrm{i}\mathbb{R} \tag{3.17}$$

  where M *is* M_- *for* $\arg k \in \left[\frac{\pi}{2}, \pi\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$, M *is* M_+ *for* $\arg k \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$, *and* J *is defined in terms of* $a, b, A, B, \mathcal{A}, \mathcal{B}$, *by equations (2.28) and (2.29).*

- $$M(x,t,k) = I + O\left(\frac{1}{k}\right) \qquad k \to \infty. \tag{3.18}$$

- *Residue conditions (2.39)–(2.43).*

Then $M(x,t,k)$ *exists and is unique. Define* $q(x,t)$ *in terms of* $M(x,t,k)$ *by*

$$q(x,t) = 2\mathrm{i} \lim_{k\to\infty} k(M(x,t,k))_{12}. \tag{3.19}$$

Then $q(x,t)$ *solves the NLS equation (1.1) with*

$$\begin{aligned} &q(x,0) = q_0(x) \qquad q(0,t) = g_0(t) \qquad q_x(0,t) = g_1(t) \\ &q(L,t) = f_0(t) \qquad q_x(L,t) = f_1(t). \end{aligned} \tag{3.20}$$

Proof. If $\alpha(k)$ and $d(k)$ have no zeros for $\arg k \in \left(0, \frac{\pi}{2}\right)$ and $\arg k \in \left(\frac{\pi}{2}, \pi\right)$ respectively, then the function $M(x,t,k)$ satisfies a non-singular RH problem. Using the fact that the jump matrix J satisfies appropriate symmetry conditions it is possible to show that this problem has a unique global solution [19]. The case where $\alpha(k)$ and $d(k)$ have a finite number of zeros can be mapped to the case of no zeros supplemented by an algebraic system of equations which is always uniquely solvable [19]. □

Proof that $q(x,t)$ *satisfies the NLS*. Using arguments of the dressing method [2], it can be verified directly that if $M(x,t,k)$ is defined as the unique solution of the above RH problem, and if $q(x,t)$ is defined in terms of M by equation (3.19), then q and M satisfy both parts of the Lax pair, hence q solves the NLS equation.

Proof that $q(x,0) = q_0(x)$. Let the 2×2 matrices $\hat{J}_1(x,k)$, $\hat{J}_3(x,k)$, $J_1^{(\infty)}(x,k)$, $J_2^{(\infty)}(x,k)$, $J_3^{(\infty)}(x,k)$ be defined by

$$\begin{aligned}
&\hat{J}_1 = \begin{pmatrix} \frac{\alpha(k)}{a(k)} & -\mathcal{B}(k)\,\mathrm{e}^{2\mathrm{i}k(L-x)} \\ 0 & \frac{a(k)}{\alpha(k)} \end{pmatrix} \qquad \hat{J}_3 = \begin{pmatrix} \frac{\overline{\alpha(\bar{k})}}{\overline{a(\bar{k})}} & 0 \\ \lambda\overline{\mathcal{B}(\bar{k})}\,\mathrm{e}^{-2\mathrm{i}k(L-x)} & \frac{\overline{a(\bar{k})}}{\overline{\alpha(\bar{k})}} \end{pmatrix} \\
&J_1^{(\infty)} = \begin{pmatrix} 1 & 0 \\ \frac{\lambda\overline{B(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kx}}{a(k)d(k)} & 1 \end{pmatrix} \qquad J_3^{(\infty)} = \begin{pmatrix} 1 & -\frac{B(k)\,\mathrm{e}^{-2\mathrm{i}kx}}{\overline{a(\bar{k})d(\bar{k})}} \\ 0 & 1 \end{pmatrix}. \\
&J_2^{(\infty)} = \begin{pmatrix} 1 & -\frac{b(k)}{\overline{a(\bar{k})}}\,\mathrm{e}^{-2\mathrm{i}kx} \\ \lambda\frac{\overline{b(\bar{k})}}{a(k)}\,\mathrm{e}^{2\mathrm{i}kx} & \frac{1}{a(k)\overline{a(\bar{k})}} \end{pmatrix}.
\end{aligned} \tag{3.21}$$

It can be verified that

$$\begin{aligned} &J_1(x,0,k) = \hat{J}_1 J_1^{(\infty)} \qquad J_2(x,0,k) = \hat{J}_1 J_2^{(\infty)} \hat{J}_3 \\ &J_3(x,0,k) = J_3^{(\infty)} \hat{J}_3 \qquad J_4(x,0,k) = J_3^{(\infty)} \left(J_2^{(\infty)}\right)^{-1} J_1^{(\infty)}. \end{aligned} \tag{3.22}$$

Let $M^{(1)}(x,t,k)$, $M^{(2)}(x,t,k)$, $M^{(3)}(x,t,k)$, $M^{(4)}(x,t,k)$ denote $M(x,t,k)$ for $\arg k \in \left[0, \frac{\pi}{2}\right], \ldots, \arg k \in \left[\frac{3\pi}{2}, 2\pi\right]$. Then the jump condition (2.27) becomes

$$M^{(2)} = M^{(1)} J_1 \qquad M^{(2)} = M^{(3)} J_4 \qquad M^{(4)} = M^{(1)} J_2 \qquad M^{(4)} = M^{(3)} J_3. \tag{3.23}$$

Evaluating these equations at $t = 0$ and using equations (3.22) we find

$$M^{(2)}(x,0,k) = (M^{(1)}(x,0,k)\hat{J}_1)J_1^{(\infty)} \qquad M^{(2)}(x,0,k) = M^{(3)}(x,0,k)J_3^{(\infty)}\big(J_2^{(\infty)}\big)^{-1}J_1^{(\infty)}$$
$$\big(M^{(4)}(x,0,k)\hat{J}_3^{-1}\big) = (M^{(1)}(x,0,k)\hat{J}_1)J_2^{(\infty)} \qquad \big(M^{(4)}(x,0,k)\hat{J}_3^{-1}\big) = M^{(3)}(x,0,k)J_3^{(\infty)}. \tag{3.24}$$

Defining $M_j^{(\infty)}(x,k)$, $j = 1,\ldots,4$, by

$$M_1^{(\infty)} = M^{(1)}(x,0,k)\hat{J}_1(x,k) \qquad M_2^{(\infty)} = M^{(2)}(x,0,k) \tag{3.25a}$$
$$M_3^{(\infty)} = M^{(3)}(x,0,k) \qquad M_4^{(\infty)} = M^{(4)}(x,0,k)\hat{J}_3^{-1}(x,k) \tag{3.25b}$$

we find that the sectionally holomorphic function $M^{(\infty)}(x,k)$ satisfies the jump conditions

$$M_2^{(\infty)} = M_1^{(\infty)}J_1^{(\infty)} \qquad M_2^{(\infty)} = M_3^{(\infty)}J_3^{(\infty)}\big(J_2^{(\infty)}\big)^{-1}J_1^{(\infty)}$$
$$M_4^{(\infty)} = M_1^{(\infty)}J_2^{(\infty)} \qquad M_4^{(\infty)} = M_3^{(\infty)}J_3^{(\infty)}.$$

These conditions are precisely the jump conditions satisfied by the unique solution of the RH problem associated with the NLS for $0 < x < \infty$, $0 < t < T$ [6]. Also $\det M^{(\infty)} = 1$ and $M^{(\infty)} = I + O\big(\frac{1}{k}\big)$, $k \to \infty$. Moreover, by a straightforward calculation one can verify that the transformation (3.25) replaces poles at ν_j by poles at k_j, with the residue conditions (2.39), (2.40), replaced by the proper residue conditions at $k = k_j$ (cf [6]). Therefore, $M^{(\infty)}(x,k)$ satisfies the same RH problem as the RH problem associated with the half-line evaluated at $t = 0$. Hence $q(x,0) = q_0(x)$.

Proof that $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$. Let $M^{(0)}(t,k)$ be defined by

$$M^{(0)}(t,k) = M(0,t,k)G(t,k) \tag{3.26}$$

where G is given by $G^{(1)},\ldots,G^{(4)}$, for $\arg k \in \big[0,\frac{\pi}{2}\big],\ldots,\big[\frac{3\pi}{2},2\pi\big]$. Suppose we can find matrices $G^{(j)}$ which are holomorphic, tend to I as $k \to \infty$, and satisfy

$$J_1(0,t,k)G^{(2)} = G^{(1)}J^{(0)} \qquad J_2(0,t,k)G^{(4)} = G^{(1)}J^{(0)} \qquad J_3(0,t,k)G^{(4)} = G^{(3)}J^{(0)} \tag{3.27}$$

where $J^{(0)}(t,k)$ is defined in remark 3.2. Then equations (3.27) yield $J_4(0,t,k)G^{(2)} = G^{(3)}J^{(0)}$, and equations (3.23) and (3.26) imply that $M^{(0)}(t,k)$ satisfies the RH problem defined in remark 3.2. Then remark 3.2 implies the desired result.

We will show that such $G^{(j)}$ matrices are

$$G^{(1)} = \begin{pmatrix} \frac{\alpha(k)}{A(k)} & c^+(k)\,\mathrm{e}^{4\mathrm{i}k^2(T-t)} \\ 0 & \frac{A(k)}{\alpha(k)} \end{pmatrix} \qquad G^{(4)} = \begin{pmatrix} \frac{\overline{A(\bar{k})}}{\overline{\alpha(\bar{k})}} & 0 \\ \lambda\overline{c^+(\bar{k})}\,\mathrm{e}^{-4\mathrm{i}k^2(T-t)} & \frac{\overline{\alpha(\bar{k})}}{\overline{A(\bar{k})}} \end{pmatrix}$$
$$G^{(2)} = \begin{pmatrix} d(k) & \frac{-b(k)}{\overline{A(\bar{k})}}\,\mathrm{e}^{-4\mathrm{i}k^2t} \\ 0 & \frac{1}{d(k)} \end{pmatrix} \qquad G^{(3)} = \begin{pmatrix} \frac{1}{\overline{d(\bar{k})}} & 0 \\ \frac{-\lambda\overline{b(\bar{k})}}{A(k)}\,\mathrm{e}^{4\mathrm{i}k^2t} & \overline{d(\bar{k})} \end{pmatrix}. \tag{3.28}$$

We recall that in the half-line problem the associated matrices $J_2^{\infty}(0,t,k)$, $G^{\infty(1)}(t,k)$, $G^{\infty(4)}(t,k)$ satisfy

$$J_2^{\infty}(0,t,k)G^{\infty(4)} = G^{\infty(1)}J^{(t)}. \tag{3.29}$$

For the verification of this equation one uses

$$a\bar{a} - \lambda b\bar{b} = 1 \qquad A\bar{A} - \lambda B\bar{B} = 1 \qquad aB - bA = \mathrm{e}^{4\mathrm{i}k^2T}c^+. \tag{3.30}$$

The matrix $J_2(0,t,k)$ can be obtained from the matrix $J_2^{\infty}(0,t,k)$ by replacing a and b with α and β; furthermore, α, β, A, B satisfy equations similar to equations (3.30) where a and b are replaced by α and β. Hence, $G^{(4)}$ and $G^{(1)}$ follow from $G^{\infty(4)}$ and $G^{\infty(1)}$ by replacing a and b by α and β; this yields the first two equations of (3.28).

Having obtained $G^{(1)}$, the first of equations (3.27) yields $G^{(2)}$ (then $G^{(3)}$ follows from symmetry considerations). Rather than deriving $G^{(1)}$ we verify that it satisfies the equation $J_1(0,t,k)G^{(2)} - G^{(1)}J^{(0)} = 0$: The (21) and (22) elements are satisfied identically. The (11) element is satisfied iff

$$\delta = \frac{\alpha}{A} + \frac{\lambda \bar{B}}{A} c^{+}\,\mathrm{e}^{4\mathrm{i}k^2 T}. \tag{3.31}$$

But

$$\delta = \frac{\alpha \bar{A} A}{A} - \lambda\beta\bar{B} = \frac{\alpha}{A}(1+\lambda B\bar{B}) - \lambda\beta\bar{B} = \frac{\alpha}{A} + \frac{\lambda\bar{B}}{A}(\alpha B - \beta A)$$

which equals the rhs of (3.31) in view of the global relation. The (12) element is satisfied iff

$$\frac{\delta b}{d\bar{A}} + \frac{\mathcal{B}\,\mathrm{e}^{2\mathrm{i}kL}}{d} = \frac{\alpha B}{A\bar{A}} - \frac{c^{+}\,\mathrm{e}^{4\mathrm{i}k^2 T}}{A\bar{A}}$$

which, using the global relation to replace $c^{+}\exp(4\mathrm{i}k^2T)$, becomes

$$\delta b + \bar{A}\mathcal{B}\,\mathrm{e}^{2\mathrm{i}kL} = \beta d. \tag{3.32}$$

The rhs of this equation is

$$b(\alpha\bar{A} - \lambda\beta\bar{B}) + \bar{A}\mathcal{B}\,\mathrm{e}^{2\mathrm{i}kL} = -\lambda\beta b\bar{B} + \bar{A}(\alpha b + \mathcal{B}\,\mathrm{e}^{2\mathrm{i}kL})$$

which equals the rhs of equation (3.32) using the identity (2.37).

Similar to the proof of the equation $q(x,0) = q_0(x)$, it can be verified that the transformation (3.26) replaces the residue conditions (2.39)–(2.43) by the residue conditions of remark 3.2.

Proof that $q(L,t) = f_0(t)$, $q_x(L,t) = f_1(t)$. Following arguments similar to the proof above we seek matrices $F^{(j)}(t,k)$ such that

$$J_1(L,t,k)F^{(2)} = F^{(1)}J^{(L)} \qquad J_4(L,t,k)F^{(2)} = F^{(3)}J^{(L)} \qquad J_3(L,t,k)F^{(4)} = F^{(3)}J^{(L)}. \tag{3.33}$$

(it is more convenient to use the second of these equations instead of $J_2(L,t,k)F^{(4)} = F^{(1)}J^{(L)}$, see below). We will show that such $F^{(j)}$ matrices are

$$F^{(1)} = \begin{pmatrix} -1 & 0 \\ \frac{-\lambda\overline{b(\bar{k})}\,\mathrm{e}^{4\mathrm{i}k^2t+2\mathrm{i}kL}}{\alpha(k)\mathcal{A}(k)} & -1 \end{pmatrix} \qquad F^{(4)} = \begin{pmatrix} -1 & \frac{-b(k)\,\mathrm{e}^{-4\mathrm{i}k^2t-2\mathrm{i}kL}}{\overline{\alpha(\bar{k})\mathcal{A}(\bar{k})}} \\ 0 & -1 \end{pmatrix}$$
$$F^{(3)} = \begin{pmatrix} -\frac{1}{\mathcal{A}(k)} & \frac{c^{+}(k)\,\mathrm{e}^{4\mathrm{i}k^2(T-t)-2\mathrm{i}kL}}{\overline{d(\bar{k})}} \\ 0 & -\mathcal{A}(k) \end{pmatrix} \qquad F^{(2)} = \begin{pmatrix} -\overline{\mathcal{A}(\bar{k})} & 0 \\ \frac{\lambda\overline{c^{+}(\bar{k})}\,\mathrm{e}^{-4\mathrm{i}k^2(T-t)+2\mathrm{i}kL}}{\mathrm{d}(k)} & -\frac{1}{\overline{\mathcal{A}(\bar{k})}} \end{pmatrix}. \tag{3.34}$$

The matrix $J_4(L,t,k)$ can be written in the form

$$J_4(L,t,k) = \overline{\Lambda(\bar{k})}\tilde{J}_4(t,k)\Lambda(k) \tag{3.35}$$

where

$$\Lambda(k) = \mathrm{diag}\left(\frac{\mathrm{e}^{2\mathrm{i}kL}}{d(k)},\ \mathrm{e}^{-2\mathrm{i}kL}d(k)\right) \tag{3.36}$$

$$\tilde{J}_4(t,k) = \begin{pmatrix} 1 & -\frac{\tilde{\beta}(k)}{(-d(k)\,\mathrm{e}^{-2\mathrm{i}kL})}\,\mathrm{e}^{-4\mathrm{i}k^2t} \\ \frac{\lambda\overline{\tilde{\beta}(\bar{k})}\,\mathrm{e}^{4\mathrm{i}k^2t}}{(-\overline{d(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL})} & \frac{1}{d(k)\overline{d(\bar{k})}} \end{pmatrix} \tag{3.37}$$

$$\tilde{\beta}(k) = A(k)b(k) - B(k)a(k). \tag{3.38}$$

Thus the second of equations (3.33) becomes

$$\tilde{J}_4(t,k)\Lambda(k)F^{(2)} = (\overline{\Lambda(\bar{k})})^{-1}F^{(3)}J^{(L)}. \tag{3.39}$$

The functions $\tilde{\beta}(k), \tilde{\alpha}(k) = -\mathrm{e}^{2\mathrm{i}kL}\overline{d(\bar{k})}, \mathcal{A}(k), \mathcal{B}(k)$ satisfy the following equations:

$$\tilde{\alpha}\bar{\tilde{\alpha}} - \lambda\tilde{\beta}\bar{\tilde{\beta}} = 1 \qquad \mathcal{A}\bar{\mathcal{A}} - \lambda\mathcal{B}\bar{\mathcal{B}} = 1 \qquad \tilde{\alpha}\mathcal{B} - \tilde{\beta}\mathcal{A} = \mathrm{e}^{4\mathrm{i}k^2T}c^+(k). \tag{3.40}$$

Indeed the first of these equations is det $\tilde{J}_4 = 1$ and the third of these equations is the global relation. Thus, comparing equation (3.39) with (3.29) it follows that $(\overline{\Lambda(\bar{k})})^{-1}F^{(3)}$ can be obtained from $G^{(\infty)(1)}$ with a, b, A, B replaced by $\tilde{\alpha}, \tilde{\beta}, \mathcal{A}, \mathcal{B}$; this yields the latter two of equations (3.34).

Having determined $F^{(2)}$, the first of equations (3.33) yields $F^{(1)}$. Rather than deriving $F^{(1)}$ we show that the equation $F^{(1)}J^{(L)} - J_1(L,t,k)F^{(2)} = 0$ is valid: the (12) and (22) elements are satisfied identically. The (21) element is satisfied iff

$$\frac{\bar{\mathcal{A}}\bar{B}\,\mathrm{e}^{2\mathrm{i}kL}}{\mathrm{d}\alpha} - \frac{a\bar{c}^+\,\mathrm{e}^{-4\mathrm{i}k^2T+2\mathrm{i}kL}}{\mathrm{d}\alpha} = \frac{\bar{b}\,\mathrm{e}^{2\mathrm{i}kL}}{\alpha\mathcal{A}} + \frac{\bar{\mathcal{B}}}{\mathcal{A}}.$$

Using the global relation to replace $\exp[-4\mathrm{i}k^2T]$, and then using $1 - a\bar{a} = -\lambda b\bar{b}$, the above equation becomes

$$\mathcal{A}(a\bar{\mathcal{B}} + \bar{b}\bar{\mathcal{A}}\,\mathrm{e}^{2\mathrm{i}kL}) = \bar{b}\,\mathrm{e}^{2\mathrm{i}kL} + \bar{\mathcal{B}}\alpha.$$

Using the definition of α, as well as $\mathcal{A}\bar{\mathcal{A}} - \lambda\mathcal{B}\bar{\mathcal{B}} = 1$, the above equation becomes an identity. The direct verification of the (11) element can be avoided by using the equality of the determinants.

Similar to the previous case, the transformation

$$M(L,t,k) \mapsto M^{(L)}(t,k) = M(L,t,k)F(t,k)$$

maps the Riemann–Hilbert problem of theorem 3.1 to the Riemann–Hilbert problem of remark 3.3.

4. The analysis of the global relation

Evaluating equation (2.17) at $x = 0$, instead of $x = 0, t = T$, we find instead of equation (1.4) the following equation:

$$\begin{aligned}(a(k)\mathcal{A}(t,k) + \overline{\lambda b(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL}\mathcal{B}(t,k))B(t,k) - (b(k)\mathcal{A}(t,k) + \overline{\lambda a(\bar{k})}\,\mathrm{e}^{2\mathrm{i}kL}\mathcal{B}(t,k))A(t,k) \\ = \mathrm{e}^{4\mathrm{i}k^2t}c^+(k,t) \qquad k \in \mathbb{C}\end{aligned} \tag{4.1}$$

where $c^+(k,t)$ has the form

$$c^+(k,t) = \int_0^L \mathrm{e}^{2\mathrm{i}kx}c(k,x,t)\,\mathrm{d}x \tag{4.2}$$

and $c(k,x,t)$ is the entire function in k which, together with its x and t derivatives, is of $O(1)$ as $k \to \infty$ and k is in the first quadrant. This in turn implies that $c^+(k,t)$ is an entire function in k which is of $O(1/k)$ as $\to \infty$, Im $k > 0$; in fact,

$$c^+(k) = O\left(\frac{1 + \mathrm{e}^{2\mathrm{i}kL}}{k}\right) \qquad k \to \infty.$$

For the analysis of equation (4.1) we will use the following two identities:

$$\int_{\partial D_1} k\left[\int_0^t \mathrm{e}^{4\mathrm{i}k^2(\tau-t')}K(\tau,t)\,\mathrm{d}\tau\right]\mathrm{d}k = \frac{\pi}{4}K(t',t) \tag{4.3}$$

$$\int_{\partial D_1^0} \frac{k^2}{\Delta(k)}\left[\int_0^t \mathrm{e}^{4\mathrm{i}k^2(\tau-t')}K(\tau,t)\,\mathrm{d}\tau\right]\mathrm{d}k$$
$$= \int_{\partial D_1^0} \frac{k^2}{\Delta(k)}\left[\int_0^{t'} \mathrm{e}^{4\mathrm{i}k^2(\tau-t')}K(\tau,t)\,\mathrm{d}\tau - \frac{K(t',t)}{4\mathrm{i}k^2}\right]\mathrm{d}k \tag{4.4}$$

where

$$t>0 \qquad t'>0 \qquad t'<t.$$

∂D_1 denotes the union of the contour $(\mathrm{i}\infty, 0]$ and of the contour $[0,\infty)$ (i.e. the oriented boundary of the first quadrant), ∂D_1^0 denotes the contour obtained by deforming ∂D_1 to the contour passing above the points $k=\frac{\pi m}{2L}$, $n\in\mathbb{Z}^+$, $K(\tau,t)$ is a smooth function of the arguments indicated and

$$\Delta(k) = \mathrm{e}^{2\mathrm{i}kL} - \mathrm{e}^{-2\mathrm{i}kL}. \tag{4.5}$$

Indeed, in order to derive equation (4.4) we rewrite the lhs of this equation as the rhs plus the term

$$\int_{\partial D_1^0} \frac{k^2}{\Delta(k)}\left[\int_{t'}^t \mathrm{e}^{4\mathrm{i}k^2(\tau-t')}K(\tau,t)\,\mathrm{d}\tau + \frac{K(t',t)}{4\mathrm{i}k^2}\right]\mathrm{d}k.$$

The integrand of the above integral is analytic and bounded in the domain of the complex-k plane enclosed by ∂D_1^0. Also its zero-order term (with respect to $(k^2)^{-1}$) contains the oscillatory factor $\mathrm{e}^{4\mathrm{i}k^2(t-t')}$, thus Jordan's lemma implies that this term vanishes. Similarly, in order to derive equation (4.3) we rewrite the lhs of this equation in the form

$$\int_{\partial D_1} k\left[\int_0^{t'} \mathrm{e}^{4\mathrm{i}k^2(\tau-t')}K(\tau,t)\,\mathrm{d}\tau\right]\mathrm{d}k + \int_{\partial D_1} k\left[\int_{t'}^{t} \mathrm{e}^{4\mathrm{i}k^2(\tau-t')}K(\tau,t)\,\mathrm{d}\tau\right]\mathrm{d}k. \tag{4.6}$$

The contour ∂D_1 involves the contour $[0,\infty)$, which can be mapped to the contour $[0,-\infty)$ by replacing k with $-k$, thus ∂D_1 can be replaced by ∂D_2 which denotes the union of the contours $(\mathrm{i}\infty,0]$ and $[0,-\infty)$. Hence replacing ∂D_1 by ∂D_2 in the first integral of the expression (4.6) we have

$$\int_{\partial\hat{D}_2}\left[\int_0^{t'} \mathrm{e}^{4\mathrm{i}k^2(\tau-t')}kK(\tau,t)\,\mathrm{d}\tau - \frac{K(t',t)}{4\mathrm{i}k}\right]\mathrm{d}k + \frac{K(t',t)}{4\mathrm{i}}\int_{\partial\hat{D}_2}\frac{\mathrm{d}k}{k}$$
$$+\int_{\partial\hat{D}_1}\left[\int_{t'}^{t} \mathrm{e}^{4\mathrm{i}k^2(\tau-t')}kK(\tau,t)\,\mathrm{d}\tau + \frac{K(t',t)}{4\mathrm{i}k}\right]\mathrm{d}k - \frac{K(t',t)}{4\mathrm{i}}\int_{\partial\hat{D}_1}\frac{\mathrm{d}k}{k}$$

where $\hat{D}$ indicates that we have indented the contour D to avoid $k=0$. The first and the third integrals vanish, since the first and the third integrands are analytic and decaying in $\hat{D}_2$ and $\hat{D}_1$ respectively, and their first-order terms (with respect to k^{-1}) contain the oscillatory factors $\mathrm{e}^{-4\mathrm{i}k^2t'}$ and $\mathrm{e}^{4\mathrm{i}k^2(t-t')}$ respectively. The remaining two integrals equal

$$\frac{K(t',t)}{4\mathrm{i}}\int_0^\pi \mathrm{i}\mathrm{d}\theta = \frac{\pi}{4}K(t',t).$$

We will now show that the global relation (4.1) can be explicitely solved for $f_1(t)$ and $g_1(t)$. To avoid routine technical complications we shall consider here the case of zero initial conditions, which yields $a(k) \equiv 1$ and $b(k) \equiv 0$. We will show that in this case the expressions for $f_1(t)$ and $g_1(t)$ are

$$\frac{\mathrm{i}\pi}{4} f_1(t) = \int_{\partial D_1^0} \frac{2k^2}{\Delta(k)} \left[\hat{M}_1(t,k) - \frac{g_0(t)}{2\mathrm{i}k^2}\right] \mathrm{d}k - \int_{\partial D_1^0} k^2 \frac{\Sigma(k)}{\Delta(k)} \left[\hat{\mathcal{M}}_1(t,k) - \frac{f_0(t)}{2\mathrm{i}k^2}\right] \mathrm{d}k + \int_{\partial D_1^0} \frac{k}{\Delta(k)} [F(t,k) - F(t,-k)]\, \mathrm{d}k \tag{4.7}$$

$$-\frac{\mathrm{i}\pi}{4} g_1(t) = \int_{\partial D_1^0} \frac{2k^2}{\Delta(k)} \left[\hat{\mathcal{M}}_1(t,k) - \frac{f_0(t)}{2\mathrm{i}k^2}\right] \mathrm{d}k - \int_{\partial D_1^0} k^2 \frac{\Sigma(k)}{\Delta(k)} \left[\hat{M}_1(t,k) - \frac{g_0(t)}{2\mathrm{i}k^2}\right] \mathrm{d}k - \int_{\partial D_1^0} \frac{k}{\Delta(k)} [\mathrm{e}^{-2\mathrm{i}kL} F(t,k) - \mathrm{e}^{2\mathrm{i}kL} F(t,-k)]\, \mathrm{d}k \tag{4.8}$$

where

$$\Sigma(k) = \mathrm{e}^{2\mathrm{i}kL} + \mathrm{e}^{-2\mathrm{i}kL} \tag{4.9}$$

and

$$F(t,k) = \frac{\mathrm{i} f_0(t)}{2} \mathrm{e}^{2\mathrm{i}kL} \hat{\mathcal{M}}_2 - \frac{\mathrm{i} g_0(t)}{2} \hat{M}_2 + \left[\overline{\hat{\mathcal{L}}_2} - \mathrm{i}\lambda \frac{f_0(t)}{2} \overline{\hat{\mathcal{M}}_1} + k \overline{\hat{\mathcal{M}}_2}\right] \left[\hat{L}_1 - \mathrm{i}\frac{g_0(t)}{2} \hat{M}_2 + k\hat{M}_1\right] - \mathrm{e}^{2\mathrm{i}kL} \left[\overline{\hat{L}_2} - \mathrm{i}\lambda \frac{g_0(t)}{2} \overline{\hat{M}_1} + k\overline{\hat{M}_2}\right] \left[\hat{\mathcal{L}}_1 - \mathrm{i}\frac{f_0(t)}{2} \hat{\mathcal{M}}_2 + k\hat{\mathcal{M}}_1\right]. \tag{4.10}$$

Indeed, substituting in equation (4.1) (with $a \equiv 1$ and $b \equiv 0$) the expressions for A, B from equations (1.12) as well as the analogous expressions for $\mathcal{A}$, $\mathcal{B}$ we find

$$-2\int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} L_1(t, 2\tau - t)\, \mathrm{d}\tau + 2\,\mathrm{e}^{2\mathrm{i}kL} \int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} \mathcal{L}_1(t, 2\tau - t)\, \mathrm{d}\tau = 2k \int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} M_1(t, 2\tau - t)\, \mathrm{d}\tau - 2k\,\mathrm{e}^{2\mathrm{i}kL} \int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} \mathcal{M}_1(t, 2\tau - t)\, \mathrm{d}\tau + \mathrm{e}^{4\mathrm{i}k^2 t} F(t,k) + \mathrm{e}^{4\mathrm{i}k^2 t} c^+(t,k). \tag{4.11}$$

Regarding $F(t,k)$ we note that we first write $F(t,k)$ in terms of $\{L_j(t,k), M_j(t,k), \mathcal{L}_j(t,k), \mathcal{M}_j(t,k)\}_1^2$ and then use equations (1.15) to rewrite $F(t,k)$ in the form (4.10). Using integration by parts it follows that $\mathrm{e}^{4\mathrm{i}k^2 t} F(t,k) = O(1/k^2)$ as $k \to \infty$.

Replacing k by $-k$ in equation (4.11) and solving the resulting equation as well as equation (4.11) for the two integrals appearing on the lhs of equation (4.11), we find the following:

$$2\int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} \mathcal{L}_1(t, 2\tau - t)\, \mathrm{d}\tau = \frac{4k}{\Delta(k)} \int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} M_1(t, 2\tau - t)\, \mathrm{d}\tau - 2k\frac{\Sigma(k)}{\Delta(k)} \int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} \mathcal{M}_1(t, 2\tau - t)\, \mathrm{d}\tau + \frac{G(t,k) - G(t,-k)}{\Delta(k)} \tag{4.12}$$

and

$$-2\int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} L_1(t, 2\tau - t)\, \mathrm{d}\tau = \frac{4k}{\Delta(k)} \int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} \mathcal{M}_1(t, 2\tau - t)\, \mathrm{d}\tau - 2k\frac{\Sigma(k)}{\Delta(k)} \int_0^t \mathrm{e}^{4\mathrm{i}k^2\tau} M_1(t, 2\tau - t)\, \mathrm{d}\tau - \frac{\mathrm{e}^{-2\mathrm{i}kL} G(t,k) - \mathrm{e}^{2\mathrm{i}kL} G(t,-k)}{\Delta(k)} \tag{4.13}$$

where

$$G(t,k) = \mathrm{e}^{4\mathrm{i}k^2 t} F(t,k) + \mathrm{e}^{4\mathrm{i}k^2 t} c^+(t,k).$$

We multiply equation (4.12) by $k\exp(-4\mathrm{i}k^2 t')$, $t' > 0$, $t' < t$, and integrate over ∂D_1^0. In this respect we note that the function

$$k\left[\frac{c^+(t,k) - c^+(t,-k)}{\Delta(k)}\right]$$

is analytic and bounded in the interior of ∂D_1^0, thus the integral of the term

$$k\,\mathrm{e}^{4\mathrm{i}k^2(t-t')}\left[\frac{c^+(t,k) - c^+(t,-k)}{\Delta(k)}\right]$$

vanishes. The integrals involving $\mathcal{L}_1$ and M_1 can be computed using equations (4.3) and (4.4), respectively. Also, the term involving $\mathcal{M}_1$ can be computed using equation (4.4) with $1/\Delta$ replaced by Σ/Δ. In this way we find that

$$\begin{aligned}
\frac{\pi}{2}\mathcal{L}_1(t, 2t'-t) &= \int_{\partial D_1^0} \frac{4k^2}{\Delta(k)}\left[\int_0^{t'} \mathrm{e}^{4\mathrm{i}k^2(\tau - t')} M_1(t, 2\tau - t)\,\mathrm{d}\tau - \frac{M_1(t, 2t'-t)}{4\mathrm{i}k^2}\right]\mathrm{d}k \\
&\quad - \int_{\partial D_1^0} 2k^2 \frac{\Sigma(k)}{\Delta(k)}\left[\int_0^{t'} \mathrm{e}^{4\mathrm{i}k^2(\tau - t')} \mathcal{M}_1(t, 2\tau - t)\,\mathrm{d}\tau - \frac{\mathcal{M}_1(t, 2t'-t)}{4\mathrm{i}k^2}\right]\mathrm{d}k \\
&\quad + \int_{\partial D_1^0} \frac{k}{\Delta(k)}\,\mathrm{e}^{4\mathrm{i}k^2(t-t')}(F(t,k) - F(t,-k))\,\mathrm{d}k.
\end{aligned}$$

Evaluating this equation at $t = t'$ and using the first and third equations of equations (1.10) and (1.11) respectively we find equation (4.7). The derivation of equation (4.8) is similar.

5. Conclusions

We have analysed the Dirichlet problem for the nonlinear Schrödinger equation on the finite interval, see equations (1.1). In particular,

(i) Given the Dirichlet data $q(0,t) = g_0(t)$ and $q(L,t) = f_0(t)$, we have characterized the Neumann boundary values $q_x(0,t) = g_1(t)$ and $q_x(L,t) = f_1(t)$ through a system of nonlinear ODEs for the functions (1.14). The functions $\{\hat{L}_j, \hat{M}_j\}_{j=1}^2$ satisfy equations (1.16), the functions $\{\hat{\mathcal{L}}_j, \hat{\mathcal{M}}_j\}_{j=1}^2$ satisfy similar equations and the Neumann boundary values are given by equations (4.7) and (4.8).

(ii) Given the initial condition $q(x,0) = q_0(x)$ we have defined $\{a(k), b(k)\}$, see definition 3.1. Given $g_0(t)$ and $g_1(t)$ we have defined $\{A(k), B(k)\}$, and given $f_0(t)$ and $f_1(t)$ we have defined $\{\mathcal{A}(k), \mathcal{B}(k)\}$, see definitions 3.2 and 3.3.

(iii) Given $\{a(k), b(k), A(k), B(k), \mathcal{A}(k), \mathcal{B}(k)\}$ we have defined a Riemann–Hilbert problem for $M(x,t,k)$ and then we have defined $q(x,t)$ in terms of M. We have shown that $q(x,t)$ solves the nonlinear Schrödinger and that

$$\begin{array}{lll} q(x,0) = q_0(x) & q(0,t) = g_0(t) & q_x(0,t) = g_1(t) \\ q(L,t) = f_0(t) & q_x(L,t) = f_1(t) & \end{array}$$

see theorem 3.1.

A general method for analysing initial-boundary value problems for integrable PDEs was introduced in [5]. This method is based on the *simultaneous* spectral analysis of the two eigenvalue equations forming the associated Lax pair, and on the investigation of the global relation satisfied by the relevant spectral functions. The rigorous implementation of this method to the NLS on the half-line was presented in [6]. Analogous results for the sine Gordon, KdV (with dominant surface tension) and modified KdV equation were presented in [7, 8]. The most difficult step of this method is the analysis of the global relation. Rigorous results for this problem were obtained in [6] by analysing the global relation, which is a scalar equation relating g_0, g_1, Φ, together with the equation satisfied by Φ, which is a vector equation relating g_0, g_1, Φ. This analysis is quite complicated and this is partly due to the fact that these two equations are *coupled*. An important development in this direction was announced in [4], where it was shown that if one uses the Gelfand–Levitan–Marchenko representation for Φ, then the above two equations can be *decoupled*. Indeed, the global relation can be *explicitly* solved for g_1 in terms of g_0 and Φ (or more precisely in terms of $\hat{L}_j$, $\hat{M}_j$, $j = 1, 2$). In this paper we have extended the results of [4, 6] to the case that the NLS is defined on a finite interval instead of the half-line.

The analysis of the analogous problem for the modified KdV equation but without using the new results of [4] is presented in [9, 10].

A different approach to this problem, which instead of the Riemann–Hilbert formalism uses the periodic extension to the whole line, is presented in [12].

For integrable evolution PDEs on the half-line, there exist particular boundary conditions for which the nonlinear Volterra integral equations can be avoided. For these boundary conditions, which we call *linearizable*, the global relation yields directly $S(k)$ in terms of $s(k)$ and the prescribed boundary conditions [6, 7]. Different aspects of linearizable boundary conditions have been studied by a number of authors, see for example [13–16]. The analysis of linearizable boundary conditions for the NLS on a finite domain will be presented elsewhere. Here we only note that *x-periodic* boundary conditions belong to the linearizable class. In this case $S(k) = S_L(k)$ and the global relation simplifies. The analysis of this simplified global relation, together with the results presented in this paper, yields a *new* formalism for the solution of this classical problem.

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